

# Several new actions of $p$ -branes based on bulk scalar fields

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**ABSTRACT:** Generally,  $p$ -branes play central roles in revealing the nonperturbative structures of the string/M-theory. In this paper, through a topological tensor current constructed in terms of a bulk scalar field, we first show some topological properties of  $p$ -branes, and then mainly discuss the construction of several new actions of  $p$ -branes. We show that the actions we construct which can be reduced to the Nambu-Goto action or some other simple actions, are all defined naturally in the bulk spacetime, preserving the full spacetime Lorentz invariance and satisfying a self-dual condition.

**KEYWORDS:**  $p$ -branes, topological tensor current, actions.

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## Contents

1. Introduction	1
2. Brief review of the $\phi$ -mapping topological tensor current theory	2
3. New actions of $p$ -branes	4
4. Properties of the new actions	8
5. Conclusion	9

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## 1. Introduction

A new idea that we may live on a brane, namely the so-called ‘brane-world’ scenario of cosmology based on string/M-theory, has been largely discussed in the last several years.  $Dp$ -branes, and for more general cases,  $p$ -branes, which are extended objects embedded in a higher dimensional bulk spacetime, have been found to play important roles in revealing the nonperturbative structures of the modern string/M-theory [1–3] and so that received much attention. As they have also been proved to be topological defects in gauge theory [4], we can study them by using some proper topological methods.

For  $p$ -branes, besides various antisymmetric tensor gauge fields living on them, in the context of the effective  $d = 10$  or  $11$  supergravity theory they are themselves  $p$ -dimensional extended sources for a  $(p + 2)$ -form gauge field strength in the bulk spacetime. Meanwhile, in the string/M-theory context one could also expect some other scalar fields, associated with many moduli fields, as fundamental fields, which, in principle, propagate in the bulk spacetime. For example, the creation of a brane world with a bulk scalar field, the effective 4-dimensional Einstein equations on the brane and a brane inflation with bulk scalar fields were discussed respectively in [5–7]. In this paper, we suppose that in the  $d$ -dimensional bulk spacetime, there is a general  $m$ -component ( $m = d - 1 - p$ ) scalar field  $\phi(x) = (\phi^1(x), \dots, \phi^m(x))$ . For general analysis, we could expect this scalar field contains many topological and dynamical information of the system. And by using a powerful tool - the  $\phi$ -mapping topological tensor current theory [8–10], we show that this is indeed a fact. Actually, through a topological tensor current constructed in terms of this field, we can not only study the relationship of  $p$ -branes and the distribution of this field, as well as some of their

topological properties, but also construct several new actions for  $p$ -branes, which govern in general the dynamics of  $p$ -branes. Since our construction is in the bulk spacetime and preserves many important symmetries naturally, surely it will be of use to many applications.

The paper is organized as follows: In Sect. 2, we give a brief and quick review of the  $\phi$ -mapping topological tensor current theory, during which we also show that  $p$ -branes are generated at the zero points of the field  $\phi(x)$  and they are quantized in terms of their winding numbers under the regular condition. In Sect. 3, based on the tensor current just obtained, we construct three new actions of  $p$ -branes, which are all defined in the bulk spacetime and through a  $\delta$ -function can all reduce to the Nambu-Goto action of  $p$ -branes. Also through introducing an auxiliary world volume metric, we show that all actions just constructed change into three other new forms. In Sect. 4, we discuss some properties of the actions we constructed, such as they preserve bulk Lorentz invariance and satisfy a self-dual condition. The conclusion and some open problems are given in Sect. 5.

## 2. Brief review of the $\phi$ -mapping topological tensor current theory

The  $\phi$ -mapping topological tensor current theory, which was proposed by Author Duan several years ago [8], has been found to be very powerful in studying topological invariants and structures of physical systems, for instances, it has been used to study the topological structures of the Gauss-Bonnet-Chern theorem, the singular structures of the London equation in superconductors, and the topological structures of the cosmic strings as well as their bifurcations, *etc* [9]. Recently, we found that besides the topological properties of physical systems, this theory also contains their dynamical properties. As it has been detailed discussed in our previous papers [10], here to be self-contained, we only give it a brief and quick review. But firstly, we give some notations used throughout this paper:  $g_{\mu\nu}$  with signature  $(-1, 1, \dots, 1)$  is the metric of the  $d$ -dimensional bulk spacetime, where the greek indices  $\mu, \nu$  run over  $0, 1, \dots, d-1$ . While,

$$h_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \quad (2.1)$$

is the induced metric in the  $(p+1)$ -dimensional world volume of  $p$ -branes spanned by  $(p+1)$  arbitrary parameters  $\xi^a$ , where the latin indices  $a, b$  run over  $0, 1, \dots, p$ . Also an auxiliary world volume metric  $\gamma_{ab}$  will be used in the next section.

By analogy with the discussion in [10], one can define a rank- $(p+1)$  topological tensor current as

$$j^{\mu_0 \dots \mu_p} = \frac{1}{A(S^{m-1})(m-1)!} \epsilon^{\mu_0 \dots \mu_p \mu_{p+1} \dots \mu_{d-1}} \epsilon_{\alpha_1 \dots \alpha_m} \partial_{\mu_{p+1}} \frac{\phi^{\alpha_1}}{\|\phi\|} \dots \partial_{\mu_{d-1}} \frac{\phi^{\alpha_1}}{\|\phi\|}. \quad (2.2)$$

where  $\|\phi\|^2 = \phi^\alpha \phi^\alpha$ ,  $\partial_\mu = \partial/\partial x^\mu$ ,  $A(S^{m-1}) = 2\pi^{m/2}/\Gamma(m/2)$  is the area of  $(m-1)$ -dimensional unit sphere  $S^{m-1}$ , and the Levi-Civita symbol  $\epsilon^{\mu_0 \dots \mu_{d-1}}$  is defined to transform as a tensor, which means that with all upper indices its components are  $\pm 1/\sqrt{-g}$  and 0 while with all lower indices its components are  $\pm \sqrt{-g}$  and 0. Obviously this topological tensor current is antisymmetric and identically conserved

$$\nabla_{\mu_i} j^{\mu_0 \dots \mu_p} = \frac{1}{\sqrt{-g}} \partial_{\mu_i} (\sqrt{-g} j^{\mu_0 \dots \mu_p}) = 0, \quad (2.3)$$

where the subscript  $i=0, \dots, d-1$ . Then just as deduced in [10], one gets a compact  $\delta$ -function structure of this tensor current rigorously

$$j^{\mu_0 \dots \mu_p} = \delta(\phi) J^{\mu_0 \dots \mu_p} \left( \frac{\phi}{x} \right), \quad (2.4)$$

where the generalized Jacobian tensor is defined as

$$\epsilon^{\alpha_1 \dots \alpha_m} J^{\mu_0 \dots \mu_p} \left( \frac{\phi}{x} \right) = \epsilon^{\mu_0 \dots \mu_p \mu_{p+1} \dots \mu_{d-1}} \partial_{\mu_{p+1}} \phi^{\alpha_1} \dots \partial_{\mu_{d-1}} \phi^{\alpha_m}, \quad (2.5)$$

and which is nontrivial only when  $\phi(x)=0$ , or equivalently

$$\phi^a(x) = 0, \quad a = 1, \dots, m. \quad (2.6)$$

This is a natural result, as in the definition (2.4) of the tensor current we have normalized the field. But in the following we will see that this current structure actually involves the total brane information of the system and indicates that all branes are located at the zero points of  $\phi(x)$ . So we will focus on these zero points.

Suppose that for the equations (2.6), there are  $K$  different solutions, namely the scalar fields  $\phi(x)$  possess  $K$  different zeros. According to the implicit function theorem, when the regular condition of  $\phi(x)$  for which the rank of the Jacobian matrix  $(\partial_\mu \phi^a)$  is  $m$  satisfies, these solutions can be expressed as

$$x^\mu = x_i^\mu(\xi^0, \dots, \xi^p), \quad i = 1, \dots, K, \quad (2.7)$$

where the subscript  $i$  represents the  $i$ th solution which is a  $(p+1)$ -dimensional submanifold spanned by the parameters  $\xi^a (a = 0, \dots, p)$  with the induced metric tensor  $h_{ab}$  and called the  $i$ th singular submanifold  $N_i$  in the bulk spacetime. For each  $N_i$  it can be proved that there exists a local  $m$ -dimensional normal submanifold  $M_i$  in the bulk spacetime spanned by the parameters  $\zeta^\alpha (\alpha = 1, \dots, m)$  with the induced metric tensor  $h'_{cd} = g_{\mu\nu} (\partial x^\mu / \partial \zeta^c) (\partial x^\nu / \partial \zeta^d)$ , which is transversal to  $N_i$  at the intersection point  $p_i$ . Then by virtue of the implicit function theorem, at the regular point  $p_i$ , the regular condition can be expressed explicitly as

$$J\left(\frac{\phi}{\zeta}\right) \equiv \frac{\partial(\phi^1, \dots, \phi^m)}{\partial(\zeta^1, \dots, \zeta^m)} \neq 0. \quad (2.8)$$

Then according to the  $\delta$ -function theory, one can expand  $\delta(\phi)$  as

$$\delta(\phi) = \sum_i \frac{\beta_i}{\left| J\left(\frac{\phi}{\zeta}\right) \right|_{p_i}} \delta(N_i) = \sum_i \frac{\beta_i \eta_i}{J\left(\frac{\phi}{\zeta}\right)|_{p_i}} \delta(N_i), \quad (2.9)$$

where  $\beta_i$  is a positive integer called the Hopf index of  $\phi$ -mapping and  $\eta_i = \text{sign} J(\phi/\zeta)|_{p_i} = \pm 1$  is the Brouwer degree, which are two topological invariant quantities.  $\delta(N_i)$  is the  $\delta$ -function on the singular submanifold  $N_i$  with the expression

$$\delta(N_i) = \int_{N_i} \delta^d(x - x_i(\xi^0, \dots, \xi^p)) \sqrt{-h} d^{p+1}\xi, \quad (2.10)$$

where  $h$  is the determinant of the induced metric  $h_{ab}$ . Substituting (2.9) into (2.4), one gets the final expansion form of the tensor current  $j^{\mu_0 \dots \mu_p}$  on the  $K$  singular submanifolds

$$j^{\mu_0 \dots \mu_p} = \sum_{i=1}^K \beta_i \eta_i \delta(N_i) \frac{J^{\mu_0 \dots \mu_p} \left( \frac{\phi}{x} \right)}{J\left(\frac{\phi}{\zeta}\right)|_{p_i}}, \quad (2.11)$$

or, in terms of parameters  $y^a = (\xi^{a_0}, \dots, \xi^{a_p}, \zeta^{b_1}, \dots, \zeta^{b_m})$ ,

$$j^{a_0 \dots a_p} = \sum_{i=1}^K \beta_i \eta_i \delta(N_i) \frac{J^{a_0 \dots a_p} \left( \frac{\phi}{y} \right)}{J\left(\frac{\phi}{\zeta}\right)|_{p_i}}. \quad (2.12)$$

Now, we see that, if taking  $\xi^0$  to be a timelike evolution parameter  $t$  and  $\xi^1, \dots, \xi^p$  spacelike parameters, as it can be proved that  $J^{\mu_0 \dots \mu_p} \left( \frac{\phi}{x} \right)/J\left(\frac{\phi}{\zeta}\right)|_{p_i}$  or  $J^{a_0 \dots a_p} \left( \frac{\phi}{y} \right)/J\left(\frac{\phi}{\zeta}\right)|_{p_i}$  has the dimension of velocity, the topological structures  $j^{\mu_0 \dots \mu_p}$  (2.11) or  $j^{a_0 \dots a_p}$  (2.12) just represent  $K$   $p$ -dimensional isolated singular objects with charges  $\beta_i \eta_i$ s moving in the bulk spacetime. These objects are just  $p$ -branes, and the  $(p+1)$ -dimensional singular submanifolds  $N_i (i = 1, \dots, K)$  are their world volumes. Furthermore, we can classify these  $p$ -branes in terms of their Brouwer degrees  $\eta_i$ : a brane is called a  $p$ -brane if its  $\eta > 0$  and an anti- $p$ -brane if its  $\eta < 0$ . Meanwhile, the product of the Hopf index and Brouwer degree of a brane can also be denoted as

$$W = \beta \eta, \quad (2.13)$$

which is an important topological quantum number - winding number - of the brane, and this just shows that  $p$ -branes are quantized in terms of their winding numbers.

### 3. New actions of $p$ -branes

We have stated that the topological tensor current also contains the dynamical properties of physical systems. Generally the dynamics of  $p$ -branes are described by their

actions. So we will study the actions of  $p$ -branes and our interest is in constructing several new actions. By using the tensor current  $j^{\mu_0 \cdots \mu_p}$  discussed in the above section, we can define an action of  $p$ -branes as

$$S_1 = - \int d^d x \sqrt{-g} \sqrt{\frac{1}{(p+1)!} g_{\mu_0 \nu_0} \cdots g_{\mu_p \nu_p} j^{\mu_0 \cdots \mu_p} j^{\nu_0 \cdots \nu_p}}, \quad (3.1)$$

in which the corresponding Lagrangian density  $\mathcal{L}$  is just a generalization of Nielsen's Lagrangian [11]. Then considering the fact that on the singular submanifolds  $N_i (i = 1, \dots, K)$ ,  $\phi(x) = 0$ , which leads to  $\partial_a \phi^\alpha(x) = \partial_\mu \phi^\alpha(x) \partial x^\mu / \partial \xi^a \equiv 0$ , it can be proved that the generalized Jacobian tensor (2.5) and scalar (2.8) have the following relation

$$J^{\mu_0 \cdots \mu_p} \left( \frac{\phi}{x} \right) = \epsilon^{a_0 \cdots a_p} \frac{\partial x^{\mu_0}}{\partial \xi^{a_0}} \cdots \frac{\partial x^{\mu_p}}{\partial \xi^{a_p}} J \left( \frac{\phi}{\zeta} \right), \quad (3.2)$$

so one gets

$$g_{\mu_0 \nu_0} \cdots g_{\mu_p \nu_p} j^{\mu_0 \cdots \mu_p} j^{\nu_0 \cdots \nu_p} = (p+1)! \delta^2(\phi) J^2 \left( \frac{\phi}{\zeta} \right), \quad (3.3)$$

in which eq. (2.4) is used. Then the action  $S_1$  simplifies to

$$S_1 = - \int d^d x \sqrt{-g} \delta(\phi) \left| J \left( \frac{\phi}{\zeta} \right) \right|. \quad (3.4)$$

Substituting (2.9) and (2.10) into it yields further that

$$S_1 = - \int d^d x \sqrt{-g} \sum_i \beta_i \int d^{p+1} \xi \sqrt{-h} \delta^d(x - x_i(\xi^0, \dots, \xi^p)) = \sum_i S_i, \quad (3.5)$$

where

$$S_i = -\beta_i \int d^{p+1} \xi \sqrt{-h} \quad (3.6)$$

is the action of the  $i$ th  $p$ -brane. Here it shows clearly that besides a coefficient difference (or if we replace  $\beta_i$  with the tension of the  $i$ th  $p$ -brane  $T_i$ ), this action is just the Nambu-Goto action of the  $i$ th  $p$ -brane. In the following, we will neglect this difference and just call (3.6) the Nambu-Goto action of the  $i$ th  $p$ -brane. Also from (3.5), one sees that the action  $S_1$  gives a unified description of all  $K$   $p$ -branes.

For a further analysis of the action  $S_1$  (3.1), as inside the square root,  $j^{\mu_0 \cdots \mu_p}$  still contains derivatives, the relative calculation based on this action should be very complicated. Actually, we can simplify it by introducing a rank- $(p+1)$  symbol

$$k^{\mu_0 \cdots \mu_p} = \epsilon^{a_0 \cdots a_p} \frac{\partial x^{\mu_0}}{\partial \xi^{a_0}} \cdots \frac{\partial x^{\mu_p}}{\partial \xi^{a_p}}, \quad (3.7)$$

which satisfies

$$k^{\mu_0 \cdots \mu_p} k_{\mu_0 \cdots \mu_p} = (p+1)!.$$

Then a new action can be defined as

$$S_2 = - \int d^d x \sqrt{-g} \frac{1}{(p+1)!} k_{\mu_0 \dots \mu_p} j^{\mu_0 \dots \mu_p}, \quad (3.8)$$

where  $k_{\mu_0 \dots \mu_p}$  now acts as a Lagrangian multiplier. By using (3.2), a similar deduction suggests that the action  $S_2$  reduces to

$$S_2 = - \int d^d x \sqrt{-g} \delta(\phi) J\left(\frac{\phi}{\zeta}\right) = \sum_i \eta_i S_i, \quad (3.9)$$

where  $S_i$  is again the Nambu-Goto action (3.6) of the  $i$ th  $p$ -brane.

If noting further that we have defined the Levi-Civita symbol  $\epsilon_{\mu_0 \dots \mu_{d-1}}$  as a tensor, which satisfies the relation

$$\epsilon_{\mu_0 \dots \mu_p} \frac{\partial x^{\mu_0}}{\partial \xi^{a_0}} \dots \frac{\partial x^{\mu_p}}{\partial \xi^{a_p}} = \epsilon_{a_0 \dots a_p},$$

(note as  $N_i$  are submanifolds in the bulk spacetime, the inverse of this relation is not correct), one gets that the action (3.8) has another equivalent but more simpler new form

$$S_3 = - \int d^d x \sqrt{-g} \frac{1}{(p+1)!} \epsilon_{\mu_0 \dots \mu_p} j^{\mu_0 \dots \mu_p}, \quad (3.10)$$

which can also reduce to eq. (3.9).

Now we have constructed three different actions of  $p$ -branes in the  $d$ -dimensional bulk spacetime by using the tensor current  $j^{\mu_0 \dots \mu_p}$ , all of which can reduce to the Nambu-Goto action of  $p$ -branes. But as is well known, actions can be constructed in several different way, and one form or another will be more useful for specific purposes. For example, the Nambu-Goto action of  $p$ -branes has geometrical meaning and is intuitively easy to understand, while the Polyakov action is very useful in the covariant quantization. In the following of this section, we will consider some other actions of  $p$ -branes in another way.

As a start, let us introduce a new independent auxiliary world volume metric  $\gamma_{ab}(\xi)$  of  $p$ -branes, which admits a covariant gauge so that simplifies the analysis and allows a covariant quantization. Henceforth ‘metric’ will always mean  $\gamma_{ab}$  (unless we specify ‘induced’), and indices will always be lowered and raised with this metric and its inverse [1]. The eq. (2.10) now changes to the form

$$\delta(N_i) = \int_{N_i} \delta^d(x - x_i(\xi^0, \dots, \xi^p)) \sqrt{-\gamma} d^{p+1} \xi, \quad (3.11)$$

and the Levi-Civita symbol  $\epsilon^{a_0 \dots a_p}$  has components  $\pm 1/\sqrt{-\gamma}$  and 0, where  $\gamma$  is the determinant of  $\gamma_{ab}$ . Also the factor  $\sqrt{-h}$  contained implicitly in the actions  $S_1$  (3.1),  $S_2$  (3.8) and  $S_3$  (3.10) is replaced by  $\sqrt{-\gamma}$ , and so we obtained another three new actions, which now are denoted as  $S'_1$ ,  $S'_2$ ,  $S'_3$  (Apparently, they have the same forms

as  $S_1$  (3.1),  $S_2$  (3.8) and  $S_3$  (3.10)). Just as the actions  $S_1$ ,  $S_2$ ,  $S_3$  can reduce to the Nambu-Goto action, we can also study the reduction of the actions  $S'_1$ ,  $S'_2$ ,  $S'_3$ . But the results will show that they correspond to different actions.

Let us first consider the action  $S'_2$ . As now one can prove

$$\begin{aligned} k_{\mu_0 \dots \mu_p} J^{\mu_0 \dots \mu_p} \left( \frac{\phi}{x} \right) &= \epsilon^{a_0 \dots a_p} \epsilon^{b_0 \dots b_p} h_{a_0 b_0} \dots h_{a_p b_p} J \left( \frac{\phi}{\zeta} \right) \\ &= -(p+1)! h \gamma^{-1} J \left( \frac{\phi}{\zeta} \right), \end{aligned} \quad (3.12)$$

in which the definition of the determinant of  $h_{ab}$  is used, it gives the action

$$S'_2 \rightarrow - \int d^d x \sqrt{-g} (-h \gamma^{-1}) \delta(\phi) J \left( \frac{\phi}{\zeta} \right) = - \sum_i \eta_i S_i, \quad (3.13)$$

where

$$S_i = -\beta_i \int d^{p+1} \xi \sqrt{-\gamma} h \gamma^{-1}. \quad (3.14)$$

This is just the Schild type (null)  $p$ -brane action [12]. So that when replace  $\sqrt{-h}$  by  $\sqrt{-\gamma}$ , through the function  $\delta(\phi)$ , the action  $S'_2$  reduce to the Schild type action of  $p$ -branes, not the Nambu-Goto action any more.

Secondly, we consider  $S'_1$ . From the properties of the Levi-Civita symbol

$$\epsilon^{a_0 \dots a_p} \epsilon_{c_0 \dots c_p} = \delta^{a_0 \dots a_p}_{c_0 \dots c_p}, \quad \epsilon^{b_0 \dots b_p} = \gamma^{b_0 c_0} \dots \gamma^{b_p c_p} \epsilon_{c_0 \dots c_p},$$

one can get

$$\begin{aligned} g_{\mu_0 \nu_0} \dots g_{\mu_p \nu_p} J^{\mu_0 \dots \mu_p} \left( \frac{\phi}{x} \right) J^{\nu_0 \dots \nu_p} \left( \frac{\phi}{x} \right) &= \epsilon^{a_0 \dots a_p} \epsilon^{b_0 \dots b_p} h_{a_0 b_0} \dots h_{a_p b_p} J^2 \left( \frac{\phi}{\zeta} \right) \\ &= (p+1)! (\gamma^{ab} h_{ab})^{p+1} J^2 \left( \frac{\phi}{\zeta} \right). \end{aligned} \quad (3.15)$$

So that the action reduces to

$$S'_1 \rightarrow - \int d^d x \sqrt{-g} (\gamma^{ab} h_{ab})^{\frac{p+1}{2}} \delta(\phi) \left| J \left( \frac{\phi}{\zeta} \right) \right| = \sum_i S_i \quad (3.16)$$

where

$$S_i = -\beta_i \int d^{p+1} \xi \sqrt{-\gamma} (\gamma^{ab} h_{ab})^{\frac{p+1}{2}}. \quad (3.17)$$

This action is just the so-called second Weyl-invariant  $p$ -brane action discussed in [12, 14], which has an important property - Weyl invariance, namely, it preserves invariance under the following Weyl transformation

$$\begin{aligned} x^\mu(\xi) &\rightarrow x^\mu(\xi), \\ \gamma_{ab}(\xi) &\rightarrow \exp(2\omega(\xi)) \gamma_{ab}(\xi), \end{aligned}$$

where  $\omega(\xi)$  is an arbitrary real function of the world volume parameters of  $p$ -branes. Further, based on this action, through a parent action, the first Weyl-invariant action can be obtained [12, 14]. And as the Weyl-invariance of actions plays a central role in the string/M-theory, the action  $S_2$  is surely worth further studies.

Still for  $S'_1$ , if using the relation

$$\begin{aligned} g_{\mu_0\nu_0} \cdots g_{\mu_p\nu_p} J^{\mu_0 \cdots \mu_p} \left( \frac{\phi}{x} \right) J^{\nu_0 \cdots \nu_p} \left( \frac{\phi}{x} \right) &= \epsilon^{a_0 \cdots a_p} \epsilon^{b_0 \cdots b_p} h_{a_0 b_0} \cdots h_{a_p b_p} J^2 \left( \frac{\phi}{\zeta} \right) \\ &= -(p+1)! h \gamma^{-1} J^2 \left( \frac{\phi}{\zeta} \right). \end{aligned} \quad (3.18)$$

which is similar to (3.12), as

$$\int d^{p+1} \xi \sqrt{-\gamma} \sqrt{h \gamma^{-1}} = \int d^{p+1} \xi \sqrt{-h}, \quad (3.19)$$

it reduces to the Nambu-Goto action of  $p$ -branes again. And this shows the classical equivalence of  $S'_1$  and  $S_1$ .

Finally, for the action  $S'_3$ , it is easy to show that it will not reduce to new actions but still gives the Nambu-Goto action through  $\delta(\phi)$ , and we will not give a repeat deduction.

## 4. Properties of the new actions

Generally the topological properties of  $p$ -branes have been shown in Sect. 2. So in this section, we mainly study their dynamical properties, namely the properties of their actions. We see that though through a  $\delta$ -function  $\delta(\phi)$ , the actions  $S_1$  (3.1),  $S_2$  (3.8) and  $S_3$  (3.10) can all reduce to the Nambu-Goto action (and  $S'_1$ ,  $S'_2$ ,  $S'_3$  reduce to some other simple actions), they have much richer structures. Firstly, from eqs. (3.5), (3.9), (3.13) and (3.16), each of the actions gives a unified description of all  $K$   $p$ -branes, which is equivalent to say that each action describes the total dynamical properties of the brane system. Also compared to the Nambu-Goto action which is defined on the world volume of  $p$ -branes, they all defined in the bulk spacetime naturally, and so that all preserve manifestly the full spacetime Lorentz invariance.

Then by using the parent action method we consider another important property of the actions (3.1), (3.8), and (3.10), and take the action  $S_2$  (3.8) as an example. The parent action approach was originally proposed to establish the equivalence or so-called *duality* between the Abelian self-dual and Maxwell-Chern-Simons models in (2+1)-dimensional spacetime, at the level of the Lagrangian instead of equations of motion, and recently it has been developed in many directions and become a powerful tool to display the duality relations of different actions [13, 14]. According to this method, we introduce two rank- $(p+1)$  tensor fields  $\Lambda_{\mu_0 \cdots \mu_p}$  and  $p^{\mu_0 \cdots \mu_p}$ , and

write down a parent action of  $S_2$

$$S_2^{parent} = - \int d^d x \sqrt{-g} \frac{1}{(p+1)!} [k_{\mu_0 \dots \mu_p} p^{\mu_0 \dots \mu_p} + \Lambda_{\mu_0 \dots \mu_p} (p^{\mu_0 \dots \mu_p} - j^{\mu_0 \dots \mu_p})], \quad (4.1)$$

where  $\Lambda_{\mu_0 \dots \mu_p}$  and  $p^{\mu_0 \dots \mu_p}$  are treated as two independent auxiliary fields. Now varying the parent action (4.1) with respect to  $\Lambda_{\mu_0 \dots \mu_p}$  gives the relation

$$p^{\mu_0 \dots \mu_p} = j^{\mu_0 \dots \mu_p}. \quad (4.2)$$

Then together with this relation, the action (4.1) turns back to the original action (3.8), and this shows the classical equivalence between the two actions. Meanwhile, if varying (4.1) with respect to  $p^{\mu_0 \dots \mu_p}$ , it leads to the relation

$$\Lambda_{\mu_0 \dots \mu_p} = -k_{\mu_0 \dots \mu_p}. \quad (4.3)$$

Substituting it into (4.1) yields again the original action (3.8). Thus we get that the action  $S_2$  (3.8) is actually a *self-dual* action. The same result also applies to the actions  $S_1$  (3.1) and  $S_3$  (3.10), and the parent actions are, respectively

$$S_1^{parent} = - \int d^d x \sqrt{-g} \left[ \sqrt{\frac{1}{(p+1)!}} p^{\mu_0 \dots \mu_p} p_{\mu_0 \dots \mu_p} + \Lambda_{\mu_0 \dots \mu_p} (p^{\mu_0 \dots \mu_p} - j^{\mu_0 \dots \mu_p}) \right] \quad (4.4)$$

$$S_3^{parent} = - \int d^d x \sqrt{-g} \frac{1}{(p+1)!} [\epsilon_{\mu_0 \dots \mu_p} p^{\mu_0 \dots \mu_p} + \Lambda_{\mu_0 \dots \mu_p} (p^{\mu_0 \dots \mu_p} - j^{\mu_0 \dots \mu_p})]. \quad (4.5)$$

(Note that to show the self-duality of  $S_1$ , it is a little more complicated than  $S_2$ , which means that after substituting the relation obtained by varying  $p^{\mu_0 \dots \mu_p}$  into the parent action (4.4) one needs a further variation in terms of  $p^{\mu_0 \dots \mu_p}$ ). Moreover, as apparently the actions  $S'_1$ ,  $S'_2$ ,  $S'_3$  have the same forms as  $S_1$ ,  $S_2$ ,  $S_3$ , the above discussion also apply to them, which is to say that  $S'_1$ ,  $S'_2$ ,  $S'_3$  are also self-dual actions.

## 5. Conclusion

In this paper, through a brief review of the  $\phi$ -mapping topological tensor current theory, we first discuss some topological properties of  $p$ -branes, and the main results are that all  $p$ -branes are generated at the zero points of the field  $\phi(x)$  and they are quantized in terms of their winding numbers. Then we discuss the construction of three new actions  $S_1$ ,  $S_2$ ,  $S_3$  of  $p$ -branes. We show that based on the results we obtained in the  $\phi$ -mapping theory and through the  $\delta$ -function  $\delta(\phi)$ , all these three actions can be reduced to the basic Nambu-Goto action of  $p$ -branes. Further, through introducing an auxiliary world volume metric  $\gamma_{ab}$ , which is used to raise and lower indices, all three actions take new forms  $S'_1$ ,  $S'_2$ ,  $S'_3$ , which then can be reduced to the

second Weyl-invariant, the Schild type and the Nambu-Goto actions, respectively. Finally, we discuss some properties of the actions that we construct. We show that compared to the Nambu-Goto action, all these actions are defined naturally in the bulk spacetime, and so preserve the full spacetime Lorentz invariance. Also, by using the parent action approach, we show that all actions we construct have an important property: they are all self-dual type actions.

Further, we have seen that during the construction of new actions, the Nambu-Goto action plays an important role. But the non-polynomial form of this action makes it difficult for canonical analysis and quantization. Other reduced actions from  $S'_1$  and  $S'_2$  also have the same or worse drawbacks. Can we construct some other actions in terms of our method, which can reduce to actions that are convenient to analyze, for example the Polyakov action or the so-called first Weyl-invariant action of  $p$ -branes? Also, we need to carry further analysis of the actions we have constructed, such as the canonical analysis, the supersymmetric extension, *etc.* They are our further works.

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